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The trapping and vertical focusing of internal waves in a pycnocline due to the horizontal inhomogeneities of density and currents

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This paper studies the propagation of a wave packet in regions where the central packet frequency ω is close to the local maximum of the effective Väisälä frequency $N_{\mathrm{ef}}(z) = N(z)/[1-\boldsymbol{k}\cdot \boldsymbol{U}(z)/\omega]$, where $\boldsymbol{k}$ is the central wavevector of the packet and $\boldsymbol{U}$ is the mean current with a vertical velocity shear. The wave approaches the layer $\omega = N_{\mathrm{ef}}^{\mathrm{m}}$ asymptotically, i.e. trapping of the wave takes place. The trapping of guided internal waves is investigated within the framework of the linearized equations of motion of an incompressible stratified fluid in the WKB approximation, with viscosity, spectral bandwidth of the packet, vertical shear of the mean current and non-stationarity of the environment taken into account. As the packet approaches the layer of trapping, the growth of the wavenumber k is restricted only by possible wave-breaking and viscous dissipation. The growth of k is accompanied by the transformation of the vertical structure of internal-wave modes. The wave motion focuses at a certain depth determined by the maximum effective Väisälä frequency $N_{\mathrm{ef}}^{\mathrm{m}}$. The trapping of the wave packet results in power growth of the wave amplitude and steepness. At larger times the viscous dissipation becomes a dominating factor of evolution as a result of strong slowing down of the packet motion.

The role of trapping in the energy exchange of internal waves, currents and small-scale turbulence is discussed.

1. Introduction

The propagation of internal waves in the horizontally inhomogeneous ocean has been intensively investigated for the last two decades. (A brief review of the papers published on this subject up to 1976 can be found in the monograph by Miropolsky (1981).) The horizontal gradients of mean density fields and flows in the ocean are, as a rule, much smaller than the vertical ones. However, the regions with significant inhomogeneity that should be taken into account are rather typical (e.g. frontal zones, synoptic eddies). On the other hand, direct satellite observations show that internal-wave packets can propagate over great distances, up to hundreds and thousands of kilometres (e.g. Apel *et al.* 1975). Even small horizontal gradients can significantly change both the kinematic and the dynamic structure of internal waves over such distances. The investigation of the interaction between internal waves and large-scale density and flow inhomogeneities is becoming increasingly relevant in view of the progress achieved in the study of the large-scale motions in the ocean that cause the horizontal inhomogeneity of mean fields (Monin, Kamenkovich & Koshlyakov 1982).

The description of internal-wave dynamics in the inhomogeneous ambient fields of density and currents is generally a very complicated problem even in a linear approximation (in a linear approximation the problem is essentially the analysis of a partial differential equation that does not allow for the separation of the variables, if vertical and horizontal inhomogeneities are taken into account simultaneously). Therefore the WKB approximation, with an assumption that the inhomogeneity is smooth in comparison with the wavelength, is generally used. Besides, a model exponential character of stratification is usually assumed (Basovich & Tsimring 1984; Miropolsky 1974; Mooers 1975*a*, *b*; Olbers 1981; Voronovich 1976*b*). The latter means that oblique propagation of three-dimensional small-scale internal-wave packets in the deep ocean is considered, with the scale of the vertical variability of the Väisälä frequency N exceeding greatly the typical wavelength (Phillips 1977). The trapping of these waves by large-scale horizontal inhomogeneities of the ambient fields was pointed out by Miropolsky (1974), Voronovich (1976*a*), Basovich & Tsimring (1984), Mooers (1975*a*) and Olbers (1981). The phenomenon of trapping consists of the asymptotic drift of the wave with intrinsic frequency Ω (in the reference system moving with the flow) to the vertical layer where the freqency Ω is equal to the local value of the constant-depth Väisälä frequency N, i.e. the wave is trapped by the layer with $\Omega = N$. It has been shown that, as the monochromatic wave approaches the layer of trapping, its amplitude tends to infinity. In the case of exponential stratification, however, the trapping is not a structurally stable effect. Olbers (1981) showed that even a minor vertical shear of the ambient flow velocity results in wave reflection at the layer $\Omega = N$.

Pronounced vertical variability of stratification and currents is the most typical feature of the upper ocean. As a result, the internal waves propagate in the upper ocean as if in a waveguide (see e.g. Phillips 1977). Smooth horizontal inhomogeneities of the ambient flow and stratification lead to gradual rearrangement in the mode and kinematic internal-wave structure. Since the inhomogeneity scales are great in comparison with the wavelength, a modification of the WKB method for multimode systems can be used to describe the internal-wave evolution (see e.g. Miropolsky 1981). This method permits one to carry out an approximate separation of the variables and to reduce the problem to a separate analysis of the vertical mode structure and slow evolution of the wave parameters along the horizontal coordinates.

In the present paper the trapping of the internal waves propagating in a waveguide is considered within the framework of such an approach. The fact that internal waves appear in the upper ocean mainly in packets makes it natural to put forward the problem of a linear description of the wave-packet trapping. If a vertical inhomogeneity of stratification is taken into account, the trapping becomes a structurally stable effect and is accompanied by a vertical focusing of the wave motion at a certain depth in the pycnocline. It is shown that a considerable growth of the wavenumber and the wave amplitude occurs in the case of internal-wave-packet trapping in real situations. As a result, the layers of trapping turn out to be regions of intense interaction between internal waves and large-scale motions.

Section 2 deals with the application of the WKB theory to the waveguide propagation of monochromatic internal waves in ambient fields that are smoothly varying horizontally. An adiabatic approximation is valid for our purposes. The trapping and the vertical focusing of monochromatic internal waves in an ideal fluid is analysed based on this approximation in §3. Different profiles of the Väisälä frequency that simulate stratification of the upper ocean are considered. In §4 we investigate the evolution of packets with finite spectral bandwidth in the vicinity of the layer of trapping, taking viscous dissipation into account. The viscous dissipation becomes a dominating factor of the packet evolution at large times because of strong slowing down of the packet velocity. Estimations of typical space–time scales of trapping are made in §5. The effect of a non-stationary mean flow is considered, and a possible role of internal-wave trapping in the energy exchange of internal waves, flows and small-scale turbulence is discussed.†

2. Description of internal-wave dynamics in the WKB approximation

The set of equations of incompressible stratified fluid dynamics linearised near the ground state $\rho = \rho_0(z, y)$, $\boldsymbol{U}_0 = \{U(z, y), 0, 0\}$ can be written in the Boussinesq approximation as

$$\left.\begin{aligned} D_t u + wU_z + vU_y &= -p_x, \\ D_t v &= -p_y, \\ D_t w + g\rho &= -p_z, \\ D_t \rho - \frac{N^2}{g} w &= -\frac{1}{\rho_0}\, \partial_y(\rho_0)\, v, \\ u_x + v_y + w_z &= 0. \end{aligned}\right\} \tag{2.1}$$

Here $\boldsymbol{u} = (u, v, w)$ are the velocity perturbations, ρ and p are the perturbations of density and pressure normalized by $\rho_0(z, y)$, $D_t = \partial_t + U\partial_x$, and $N = [(g/\rho_0)\,\partial_z \rho_0]^{\frac{1}{2}}$ is the Brunt–Väisälä frequency.

The ground-state parameters $\rho_0(z, y)$ and $U(z, y)$ are related by the equations of motion. In the real ocean, however, the horizontal variability of the currents has a much greater influence on the internal-wave dynamics than the horizontal variability of the density field. Nevertheless, the horizontal variation of statification can sometimes be a predominant factor (e.g. for waves propagating nearly perpendicularly to the flow $U(z, y)$). Therefore the effects of these two factors will be analysed separately. The relation between $\rho_0(z, y)$ and $U(z, y)$ that are, for example, related geostrophically

$$\frac{f_0}{g}\,(U\rho_0)_z = \rho_{0y} \tag{2.2}$$

(where f_0 is the Coriolis parameter) can be easily taken into account.

Let us first consider the propagation of monochromatic waves of the

$$f \sim f(y, z)\exp\{\mathrm{i}k_x x - \mathrm{i}\omega t\}$$

type in an inviscid fluid. Here ω and k_x are respectively the frequency and the x-component of the wavevector. It seems more convenient to rewrite the set of equations (2.1) in the form of an equation for one of the dependent variables, e.g. for the pressure p:

$$(U-c)^2\,\partial_y\{(U-c)^2\,\partial_y p\} + \partial_z\{B\,\partial_z p\} + k_x^2 p + \frac{g\,\partial_z\{B\,\partial_y p\,\partial_y \rho_0\}}{\rho_0 k_x^2 (U-c)^2} = 0, \tag{2.3}$$

† A short communication on this work has been published in the Reports of the Academy of Sciences of the USSR (Badulin, Tsimring & Shrira 1983).

where
$$c = \frac{\omega}{k_x}, \quad B = \left\{1 - \frac{N^2(z,y)}{k_x^2(U-c)^2}\right\}^{-1}.$$

The basic assumption used in this paper is that of smooth variability of the ambient fields U and N along the horizontal y-coordinate. Following Voronovich (1976), we introduce formally a small parameter ϵ equal to the ratio of the typical wavelength λ_0 and the characteristic scale of the horizontal inhomogeneity D. Here D is also much greater than the characteristic vertical scale of stratification d. Hence we can seek a solution in the form

$$p(z,y) = p(z,\epsilon y)\exp\left\{\mathrm{i}\int k_y(\epsilon y)\,\mathrm{d}y\right\}. \tag{2.4}$$

For simplicity we consider first a vertically homogeneous flow $U(y)$. Neglecting the terms of order ϵ^2 in (2.3), we obtain

$$\partial_z[B\,\partial_z p] - (k_x^2 + k_y^2)\,p = \epsilon\left\{2\mathrm{i}k_y\frac{p\,\partial_y U}{U-c} - \frac{g}{\rho_0}\frac{\mathrm{i}k_y\,\partial_z[Bp\,\partial_y\rho_0]}{k_x^2(U-c)}\right\}. \tag{2.5}$$

Away from the vicinity of the singular points ($B \rightarrow \infty$, $U \rightarrow c$) the terms in the right-hand side are of order ϵ. In the zeroth-order approximation, the right-hand side of (2.5), i.e. the terms of order ϵ, is neglected. It is convenient to rewrite the equation of the zero or adiabatic approximation for the vertical velocity component w, taking into account that

$$w = \frac{B\,\partial_z p}{\mathrm{i}k_x(U-c)} - \frac{g}{\rho_0}\partial_y\rho_0\frac{B\,\partial_y p}{k_x^3(U-c)^3}. \tag{2.6}$$

In the zeroth-order approximation, (2.5) and (2.6) yield

$$\partial_{zz}w - \left\{1 - \frac{N^2(z,\epsilon y)}{k_x^2(U-c)^2}\right\}[k_x^2 + k_y^2(\epsilon y)]\,w = 0. \tag{2.7}$$

Together with standard boundary conditions at the bottom and on the surface, (2.7) is a boundary-value problem. Solving (2.7), one finds the wavenumber k and the dependence of the eigenfunction $w(z)$ on ω, U and N. The wavenumber and the eigenfunctions, as well as U and N, parametrically depend on y. The dispersion relation can be written conventionally

$$\Omega = \Omega(k_x, k_y(y)), \quad \Omega = \omega - k_x U. \tag{2.8}$$

The use of the adiabatic WKB approximation, i.e. the parametric dependence of the dispersion relation for internal waves and the mode structure on y, means the application of the relations between the internal-wave parameters of the horizontally homogeneous fluid for a smoothly inhomogeneous case. The validity of the zeroth-order WKB approximation, as the internal waves approach the layer of trapping (where $B \rightarrow \infty$), is discussed in §4.4.

3. The trapping and the vertical focusing of a monochromatic internal wave in an ideal fluid

3.1. *The main characteristics of trapping*

We define *the layer of trapping as a vertical plane that is asymptotically approached by a narrow-spectrum packet.*

We can show that, for a monochromatic wave with Doppler frequency Ω, the vertical plane where Ω coincides with the local maximum of the Väisälä frequency N_m

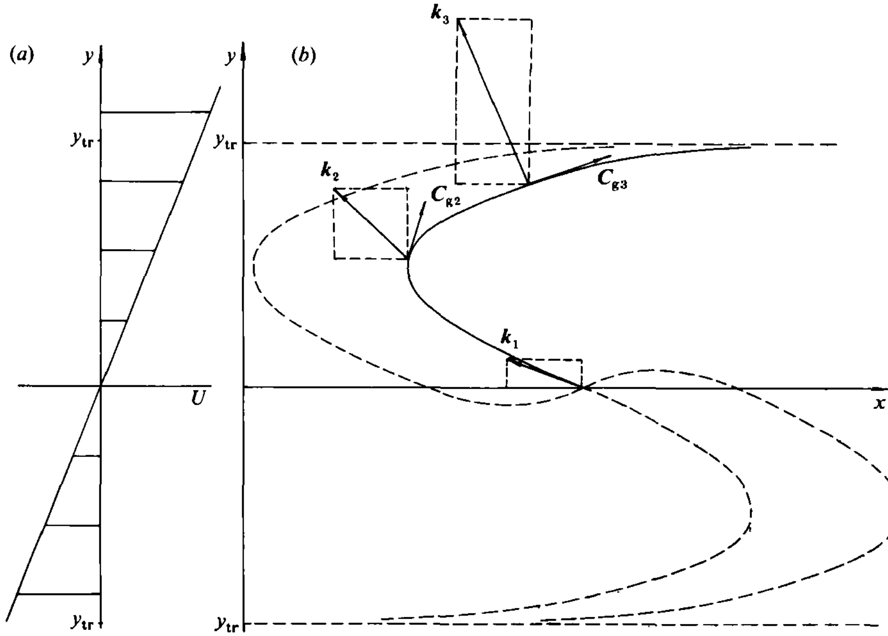

FIGURE 1. The trajectories of the packet trapped by the flow inhomogeneity (N_m = const, $U = U'_y(0)\,y$). (*a*) The current profile. (*b*) The trajectories of four packets of the same (modulo) initial (at $y = 0$) frequency and wavevector $\boldsymbol{k}$ projections (k_x, k_y). The solid line corresponds to one of the two identical cases (antisymmetrical curves) when $\boldsymbol{k}\,\partial_y(\boldsymbol{U})\,|_{y=0}$ is negative. The unlimited monotonic increase of k_y (while k_x remains constant) and decrease of c_{gy} (the y-component of the group velocity c_g) is demonstrated by showing $\boldsymbol{k}$ and $\boldsymbol{c}_g$ at three points (numbered 1, 2, 3) of the trajectory. Near the layer of trapping, c_{gy} tends to zero; the difference $c_{gx} - U(y = y_{tr})$ also tends to zero. Two of the dashed trajectories correspond to packets with positive initial value of $\boldsymbol{k}\,\partial_y(\boldsymbol{U})$. These packets, before the same process of trapping begins, should first pass over the simple reflection point (where c_{gy} and k_y change signs).

is the layer of trapping. Let the zero value of y correspond to this layer. Using the y-power expansion of $\Omega^2(y)/N_m^2(y)$ in the vicinity of this plane

$$\frac{\Omega^2(y)}{N_m^2(y)} = 1 - \alpha y, \tag{3.1}$$

and the dispersion relation in the form

$$\Omega^2 = N_m^2(1 - f(k)), \tag{3.2}$$

we obtain

$$f(k) = \alpha y. \tag{3.3}$$

It follows from the general properties of the dispersion curves for internal waves (see e.g. Phillips 1977; Miropolsky 1981) that when $y \to 0$ then $f(k(y)) \to \infty$, $k(y) \to \infty$. We can easily calculate the time τ_* required for the narrow internal-wave packet to reach a certain point y_* that belongs to the region of applicability of (3.1) and (3.3):

$$\tau_* \approx \frac{1}{N\alpha}(k_* - k_0), \tag{3.4}$$

where k_* is the value of the wavenumber at the point y_* and k_0 ($\ll k_*$) is the initial value of the wavenumber. It follows from (3.4) that an *infinite time* is required for

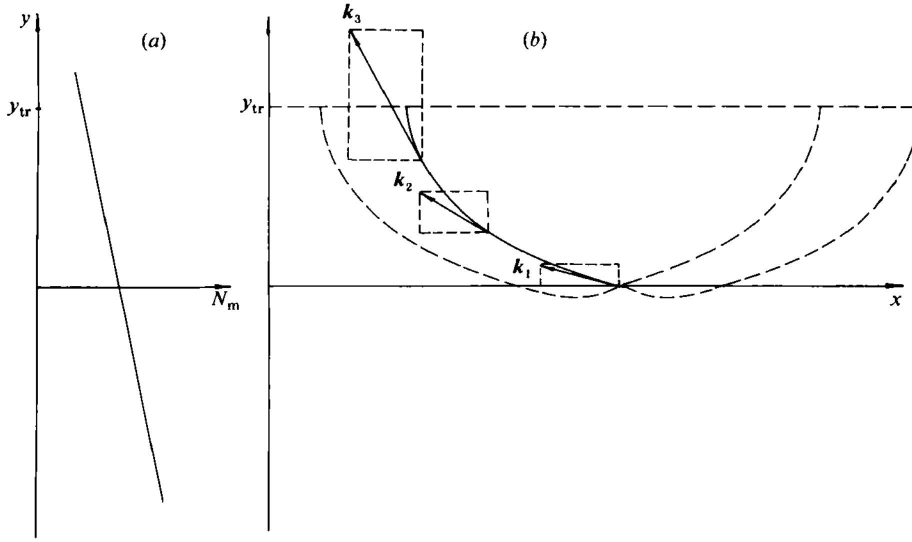

FIGURE 2. The trapping by the inhomogeneity of stratification ($U = 0$, N_m = const). (*a*) The $N_m(y)$ dependence. (*b*) The trajectories of four packets of the same (modulo) initial frequency and wavevector $\boldsymbol{k}$ projections. The solid line corresponds to one of the two symmetrical trajectories with negative initial $k_y \partial_y N$. The unlimited monotonic increase of k_y (while k_x is constant) is demonstrated by showing $\boldsymbol{k}$ at points 1, 2, 3 of the trajectory. As the direction of $\boldsymbol{c}_g$ coincides with $\boldsymbol{k}$, changes of $\boldsymbol{c}_g$ are not shown. Two symmetrical dashed trajectories (with $k_y \partial_y N$ positive) correspond to the packets that should first pass over the simple reflection point before the same process of trapping begins. The packets with the same ω and $|k_x|$ are trapped by the same layer of trapping.

the internal-wave packet to reach the point $y = 0$. Thus the layer $\Omega = N_m$ is the layer of trapping. This fact accounts for rather strong dissipative, dispersive and nonlinear effects in the vicinity of the layer. The picture of the packet kinematics is sketched in figures 1 and 2, for the cases of linear horizontal inhomogeneities of the mean flow and stratification respectively. These figures, in particular, demonstrate that a packet is *always trapped* if horizontal gradients of U and N can be considered constant.

In order to find the internal-wave amplitude, we use the equation for the conservation of the wave action, or, to be more exact, its stationary form (see Voronovich 1976*b*)

$$A^2 c_g \int_0^h \frac{N^2(z)\, w^2}{\Omega^3}\, dz = \text{const}, \tag{3.5}$$

where $w(z)$ are the eigenfunctions of (2.7) taken in the 'rigid-lid' approximation. It should be noted that the conservation law (3.5) follows from the condition that the terms on the right-hand side of (2.5) are orthogonal to the eigenfunctions, i.e. to the zeroth-order solution. It is readily seen from (3.5) that, as the wave approaches the layer of trapping ($y \to 0$), its displacement amplitude A tends to infinity.

There are two mechanisms that lead to the growth of the wave amplitude in the vicinity of the layer of trapping. The first one is associated with the 'slowing down' of the internal-wave packet, when $c_g \to 0$ at $y \to 0$. The second one is due to the maximum in the vertical profile of the Väisälä frequency $N(z)$: as the packet approaches the layer of trapping, the wave motion tends to concentrate in the vicinity of the maximum N, where $\Omega < N(z)$. Since the width of the localization region decreases, as the layer of trapping is approached, the wave amplitude also increases due to the vertical focusing (see figure 3).

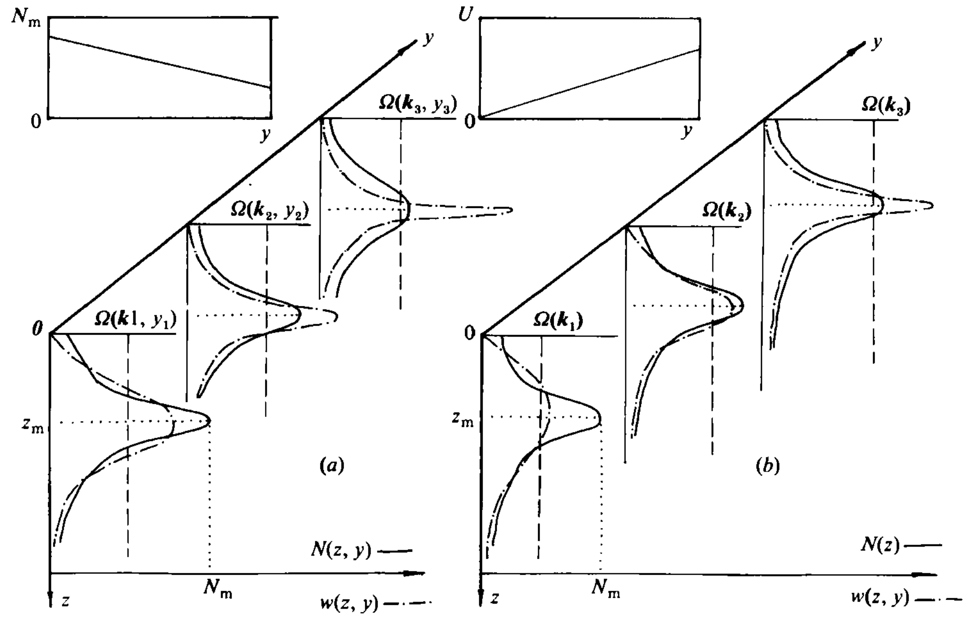

FIGURE 3. The vertical focusing of the wave trapped by inhomogeneity (*a*) of stratification ($N_m(y)$ is shown in the box) and (*b*) of current ($U(y)$ is shown in the box). The profiles of $N(z)$ and of the first-mode eigenfunction $w(z)$ are plotted. Points 1, 2, 3 correspond to the points 1, 2, 3 of the trapped-packet trajectory (see figures 1 and 2). The dashed lines mark the value of the intrinsic packet frequency $\Omega(\boldsymbol{k})$.

3.2. *Trapping in a model stratification profile*

Consider as an example a model density distribution in the hyperbolic-tangent form that qualitatively approximates typical oceanic profiles. Assume that

$$N^2 = \frac{N_m^2}{\mathrm{ch}^2\,[2d^{-1}(z-h)]}, \tag{3.6}$$

where d is the characteristic scale of varying stratification. Thus (2.7) can readily yield a dispersion relation, the expression for the group velocity c_g and the eigenfunctions $w_h(z)$ of the boundary-value problem (Krauss 1965)

$$\Omega^2 = N_m^2 \left\{1+\frac{2(2n+1)}{kd}+\frac{(2n+1)^2-1}{(kd)^2}\right\}^{-1}, \tag{3.7a}$$

$$c_g = N_m d \left\{2\left[1+\frac{2(2n+1)}{kd}+\frac{(2n+1)^2-1}{(kd)^2}\right]\right\}^{-\frac{3}{2}} \left\{\frac{2(2n+1)}{(kd)^2}+\frac{2[(2n+1)^2-1]}{(kd)^3}\right\}, \tag{3.7b}$$

$$w = c_1\,\xi^{-\frac{1}{2}m}\,F(-\tfrac{1}{2}n, \tfrac{1}{2}(m+2n+1), m+1, \xi^{-1}). \tag{3.8}$$

Here F is the hypergeometric function, $\xi = \mathrm{ch}^2\,(2z-h)/d)$, $m = \frac{1}{2}kd$. For the fundamental mode $n = 0$, $w = c_1\,\xi^{-\frac{1}{2}m}$. Assuming the normalization condition to be $\int_{-\infty}^{+\infty} N^2 w^2\,\mathrm{d}z = 1$, we find the constant c_1. Equations (3.7) and (3.8) together with the conservation law (3.5) completely determine the variation in the wave parameters. Assuming that $kd \gg 1$ near the layer of trapping, we can use asymptotic forms of (3.7) and (3.8) at $m \gg 1$. The asymptotic form of the wave-displacement amplitude A can be easily found by some simple transformations:

$$A \sim (kd)^{\frac{5}{4}} \sim \left(\frac{y}{D}\right)^{-\frac{5}{4}} \sim (Nt)^{\frac{5}{4}}. \qquad (3.9)$$

Note that neither the fact that the wave amplitude tends to infinity nor the asymptotic growth rate as the packet approaches the layer of trapping depend on the chosen model profile $N(z)$. The reasons for the universal behaviour of the wave in the vicinity of the layer of trapping can be clearly seen from the analysis of the problem given below.

3.3. *The asymptotic behaviour of the trapped internal waves*

Let us consider the case where the wave frequency is, from the very beginning, close enough to the maximum Väisälä frequency. The mode is then strongly localized in the vicinity of the maximum $N_{\rm m}$, and the expansion of N in the z-power series can be used

$$N^2 = N_{\rm m}^2\left(1-\frac{z^2}{d^2}\right). \qquad (3.10)$$

Such an expansion is justified for any smooth profile $N(z)$. Thus (2.7) reduces to

$$\partial_{zz} w + \left\{k^2\left(\frac{N_{\rm m}^2}{\Omega^2}-1\right)-k^2\,\frac{N_{\rm m}^2}{\Omega^2}\,\frac{z^2}{d^2}\right\} w = 0.$$

The boundary-value problem can be easily solved assuming that, since the mode is strongly localized in the vicinity of $N_{\rm m}$, the boundary conditions can be taken in the form $|w(z)|\rightarrow 0$, when $|z|\rightarrow\infty$. Then the solution is

$$w(z,y) = H_n[\kappa_n(y)\,z]\exp\{-\tfrac{1}{2}[\kappa_n(y)\,z]^2\}, \qquad (3.11)$$

where H_n is the Hermite polynomial, $\kappa_n(y) = (kN_{\rm m}/d\Omega)^{\frac{1}{2}}$, and $k_n(y)$ is the eigenvalue of the corresponding boundary-value problem

$$k_n(y) = \frac{2n+1}{d}\left(\frac{N_{\rm m}}{\Omega}-\frac{\Omega}{N_{\rm m}}\right)^{-1}.$$

For the fundamental mode ($n = 0$), we have

$$k(y) = \left[d\left(\frac{N_{\rm m}(y)}{\Omega(y)}-\frac{\Omega(y)}{N_{\rm m}(y)}\right)\right]^{-1}, \quad w(z,y) = \frac{1}{\pi^{\frac{1}{2}}}\exp\left\{-\frac{k(y)\,N_{\rm m}}{2\Omega d}\,z^2\right\}. \qquad (3.12)$$

It is easily seen from (3.11) and (3.12) that the characteristic vertical scale $d_1 = (d/k)^{\frac{1}{2}}$ of the eigenfunction variation is much smaller than d when $kd \gg 1$. Thus the expansion (3.10) for short internal waves is justified. It also follows from (3.11) and (3.12) that, when the wave approaches the layer of trapping, the wavenumber increases inversely to y, and the vertical scale of the mode decreases proportionally to $y^{\frac{1}{2}}$. Using the conservation of the wave action in the form (3.5) and the formulae

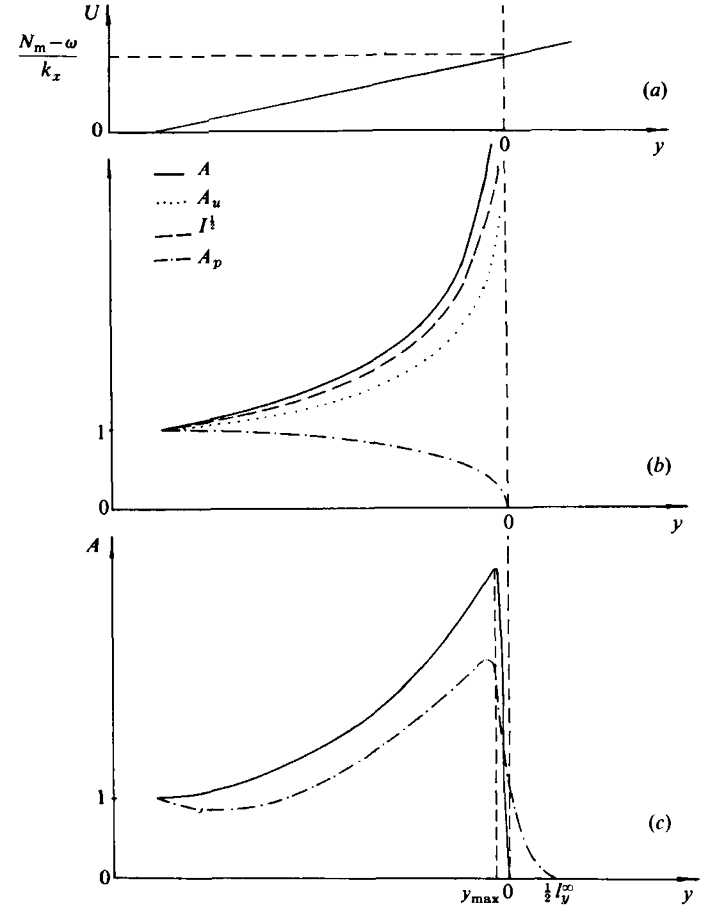

FIGURE 4. The growth of the packet amplitude near the layer of trapping. (*a*) Sketch of current geometry. (*b*) The quasimonochromatic wave in an inviscid fluid. $A \equiv$ wave amplitude of displacement; $A_u \equiv$ horizontal velocity amplitude; $I \equiv$ wave action ($I^{\frac{1}{2}}(y)$ dependence shows the rate of amplitude A growth due to the 'slowing down' only, not taking into account the vertical focusing), $A_p \equiv$ pressure-variation amplitude. Near $y = 0$, $A \sim y^{-\frac{5}{4}} \sim t^{\frac{5}{4}}$, $A_u \sim y^{-\frac{3}{4}} \sim t^{\frac{3}{4}}$, $I^{\frac{1}{2}} \sim y^{-1} \sim t$, $A_p \sim y^{\frac{1}{4}} \sim t^{-\frac{1}{4}}$ (for smooth $N(z)$ profiles). (*c*) Solid line shows $A(y)$ ($A \equiv$ displacement amplitude of a monochromatic wave) with viscosity taken into account. Dot–dashed line shows amplitude $A(y)$ with both viscosity and finite bandwidth taken into account.

(3.12), and assuming that the profile $\Omega(y)/N_m(y)$ in the vicinity of $y = 0$ is linear, (3.1), we obtain the law of wave-amplitude growth

$$A^2 = \frac{\text{const}}{N_m d/k} \frac{N_m^2}{\pi^{\frac{1}{2}}} \left[\left(\frac{\Omega d}{k N_m} \right)^{\frac{1}{2}} - \frac{1}{2d^2} \left(\frac{k N_m}{\Omega d} \right)^{-\frac{3}{2}} \right], \tag{3.13}$$

$$A \sim (kd)^{\frac{5}{4}}, \quad A_u \sim (kd)^{\frac{3}{4}}, \tag{3.14}$$

which coincides with the asymptotic behaviour of the problem with model stratification (3.6). Here A is the wave-displacement amplitude, and A_u is the horizontal velocity amplitude (see figures 4*a*, *b*).

3.4. *Non-smooth stratification profiles*

The universality of the asymptotic forms (3.14) is restricted by the assumption that the profile $N(z)$ is smooth in the vicinity of the maximum and can be represented by (3.10). Consider a more general case of simulating stratification in the pycnocline region. Let

$$N(z) = N_m(1-a|z|^{\beta}), \quad a = \begin{cases} a_1 & (z > 0), \\ a_2 & (z < 0). \end{cases} \tag{3.15}$$

Then the dispersion relation in the short-wave limit has the form (Dotsenko 1982)

$$\Omega^2 = N_m^2(1-kd)^{-\gamma}), \quad \gamma = \frac{2\beta}{\beta+2} \quad (0 < \gamma \leqslant 2). \tag{3.16}$$

It is seen from the boundary-value problem (2.7) that when k is large enough the width of the region of vertical localization decreases with the growth of k as $k^{-\gamma/\beta}$. From conservation of wave action in the form (3.5) we obtain the law of wave-amplitude evolution using the expansion (3.1)

$$A \sim (kd)^{\frac{1}{4}(4+\gamma)} \sim (\alpha y)^{-(4+\gamma)/4\gamma} \sim (Nt)^{\frac{1}{4}(4+\gamma)}. \tag{3.17}$$

Thus, for a more general form of the stratification profile, the amplitudes of displacement and velocity also tend to infinity when the packet approaches the layer of trapping.

3.5. *The effect of stable flow velocity shear*

Let us now consider the effect of the vertical shear in the flow velocity. Assume that the shear flow is stable and there are no critical layers: $Ri = (N/U_z')^2 \gg \frac{1}{4}$, $\omega - k_x U(z, y) \neq 0$. Then the generalization of the boundary-value problem (2.7) takes the form (Voronovich 1976*a*; Miropolsky 1982)

$$\partial_{zz} w + \left\{ \frac{N^2(z,y)}{(\omega - k_x U(z,y))^2} + \frac{k_x U_{zz}''}{k^2(\omega - k_x U(z,y))} - 1 \right\} k^2 w = 0. \tag{3.18}$$

Using the stability condition of the flow, we can assume that in the vicinity of the layer of trapping $U_{zz}'' \ll k^2N^2/(\omega - k_x U) k_x \sim k^2N/k_x$. Then neglecting the term $\sim U_{zz}''$ in (3.18) and *introducing the effective Väisälä frequency* $N_{ef} = N(z)/(1 - k_x U/\omega)$, *we completely reduce the problem with* $U_z' \neq 0$ *to the one considered previously.* Thus the results presented in this section are also valid for shear flows. Note that N_{ef} depends strongly on the direction of wave propagation. Different waveguides and horizons of localization correspond to the waves propagating in different directions in the flow with arbitrary $U(z)$. In some directions the trapping does not take place.

4. The effect of viscosity on internal-wave packet evolution with a finite spectral bandwidth in the vicinity of the layer of trapping

It was shown above that in an ideal fluid the amplitudes of monochromatic waves, as they approach the layer of trapping, tend to infinity according to a power law. Let us estimate the limitations set on the growth of the internal-wave amplitude due to viscosity and the bandwidth of the wave-packet spectrum. We first consider the effect of viscosity on monochromatic waves, and then the influence of dispersion on inviscid and viscous evolution of the wave packets.

4.1. *Monochromatic internal waves*

Let us study the effect of viscosity on monochromatic waves. In order to take viscous dissipation into account, we should start from the Navier–Stokes equations. Following a procedure similar to that described in §2, we obtain a viscous analogue of the boundary-value problem (2.7):

$$\partial_{zz} w - \frac{\left[\left(\frac{N^2}{\Omega^2}-1\right)k^2+\frac{\mathrm{i}\nu k^4}{\Omega}\right]}{1-\frac{2\mathrm{i}\nu k^2}{\Omega}} w + \frac{\mathrm{i}\,\partial_{zzzz} w}{\frac{\Omega}{\nu}-2\mathrm{i}k^2} = 0. \tag{4.1}$$

It is natural to consider the viscosity to be small, i.e. to assume the frequency correction caused by viscosity to be small: $\Gamma = \nu k^2/\Omega \ll 1$. (This is justified when $k^2 \ll N_m/\nu$, i.e. for *the whole range* of the internal-wave scales in the ocean.) Since the viscosity is small, we can express $\partial_{zzzz} w$ using the solution of (4.1) without this term, and substitute the latter into (4.1):

$$\partial_{zz} w + \left\{\frac{\left(\frac{N^2}{\Omega^2}-1\right)k^2+\frac{\mathrm{i}\nu k^4}{\Omega}}{1-\frac{2\mathrm{i}\nu k^2}{\Omega}} + \frac{\mathrm{i}\nu}{\Omega}\left[\frac{\left(\frac{N^2}{\Omega^2}-1\right)k^2+\frac{\mathrm{i}\nu k^4}{\Omega}}{1-\frac{2\mathrm{i}\nu k^2}{\Omega}}\right]^2\right\} w = 0. \tag{4.2}$$

Restricting our consideration to the first order in Γ, we can obtain a viscous modification of the dispersion relation from (4.2):

$$\Omega \approx \Omega_{\nu=0}(k)\left(1+\mathrm{i}\frac{\Omega}{N}\frac{\nu k^2}{2\Omega}\right). \tag{4.3}$$

The solutions of the boundary-value problem (2.7) (neglecting viscosity) can be used if the conditions $\Gamma \ll 1$ and

$$\Gamma \ll \frac{N^2}{\Omega^2}-1 \tag{4.4}$$

are fulfilled.

The fact that the viscosity is small allows us to find easily the transformation of the wave action I.

For a stationary case

$$\partial_y(c_{gy} I) + 2\nu k^2 I = 0. \tag{4.5}$$

Making use of the fact that according to (3.3) $\mathrm{d}y = (c_g D/N_m)\,\mathrm{d}k$ in the vicinity of the layer of trapping and neglecting the terms $\sim k_x^2/k^2$, we obtain

$$I = I_0 \frac{c_{gy}(y_0)}{c_{gy}(y)} \exp\left\{-\int_{y_0}^{y} \frac{2\nu k^2(y)\,\mathrm{d}y}{c_{gy}(y)}\right\} \approx \frac{I_0}{c_{gy}/c_{gy}^0} \exp\left\{-\frac{4\nu k^3}{3}\frac{D}{N_m}\right\}. \tag{4.6}$$

From (4.6) we find the dependences of the wave steepness and of the displacement and horizontal velocity amplitudes on k and y. The dependences have a qualitatively similar character. Then, after the parameter maximum has been reached (note that the maximum of each parameter occurs at its particular value of k_{max} and y_{max}), an extremely abrupt decay takes place ($\sim \exp(-k^3/k_0^3)$). Thus the dissipation sets an effective lower limit on the range of the horizontal and hence vertical wavescales.

Let us now consider the evolution of the wave-displacement amplitude for the stratification model (3.15):

$$\frac{A^2}{A_0^2} \sim (kd)^{\frac{1}{4}(4+\gamma)} \exp\left(-\frac{4\nu k^3 D}{3N_m}\right). \tag{4.7}$$

The maximum of A is reached when $k = k^{(A)}_{\mathrm{max}}$, where

$$k^{(A)}_{\mathrm{max}} = \left[(1+\tfrac{1}{4}\gamma)\, \frac{N_{\mathrm{m}}}{\nu D} \right]^{\frac{1}{3}}. \tag{4.8}$$

It was mentioned above that the values of the stratification parameter γ lie in the interval $(0,2)$, and $\gamma = 1$ corresponds to the smooth profiles $N(z)$ of the form (3.10). Direct substitution makes it obvious that when $k \sim k^{A}_{\mathrm{max}}$ the condition (4.4) is fulfilled (the ratio $\Gamma/(N^2/\Omega^2 - 1) \sim d/D$). *Thus viscous dissipation remains small even in the region where its influence on the wave evolution is dominating.* Such a pronounced influence of small viscosity can be accounted for by the drastic decrease in the packet group velocity, and hence by the packet slowing down. The time required for the packet to pass a distance of the wavelength along y is much larger than $(\nu k^2)^{-1}$. For a model stratification with the profile $N(z)$ in the form of a hyperbolic secant (3.6), the range of parameters typical of the ocean ($d \sim 10^4$ cm, $D \sim 10^6$–10^8 cm, $N_{\mathrm{m}} \sim 10^{-3}$–$10^{-2}\ \mathrm{s}^{-1}$, $\nu \sim 10^{-2}$–$10^{-1}\ \mathrm{cm}^2\,\mathrm{s}^{-1}$) and for the value $\Omega_0 = N_{\mathrm{m}}/\sqrt{2}$, we obtain the estimates $k^{A}_{\mathrm{max}} \sim 10^{-2}$–$10^{-3}\ \mathrm{cm}^{-1}$ and $A_{\mathrm{max}}/A_0 \sim 10$–$10^2$.

4.2. *The effect of finite spectral bandwidth*

Let us consider the influence of the packet spectral bandwidth. The evolution of packets with an infinitely narrow spectrum (i.e. non-dispersive packets) has been described above. The finiteness of the bandwidth spectrum leads to considerable dispersive spreading of the packet in two horizontal directions. The dynamics of a packet with finite spectral bandwidth can be described correctly by means of the Fourier integral taken over harmonic components. Let us, however, use a simpler method for the estimation of the dispersive effect. By virtue of the conservation law (3.5), the integral of the wave action over the whole region occupied by the field is retained. Let us describe the field region (i.e. the horizontal dimensions of the packet) by the values l_x and l_y, and the wave field by the value of the wave action $\bar{I}$ averaged over the field region. The integral characteristics $\bar{I}$, l_x and l_y at a given point (x, y) are related to the initial values as follows:

$$c^0_{\mathrm{g}}(x_0, y_0)\, l^0_x\, l^0_y\, \bar{I}^0 = \bar{c}_{\mathrm{g}}(x, y)\, l_x\, l_y\, \bar{I}(x, y). \tag{4.9}$$

The horizontal dimensions of the packet with a central frequency ω_{c} and a spectral width $\Delta\omega$ grow with time according to the law

$$l_y = l^0_y + \int_0^t [c_{\mathrm{g}y}(k_{\mathrm{c}} - \Delta k) - c_{\mathrm{g}y}(k_{\mathrm{c}} + \Delta k)]\,\mathrm{d}t, \quad l^0_y = \frac{2\pi}{\Delta k^0_y} \approx \frac{2\pi}{\omega}\, c^0_{\mathrm{g}y}(k_{\mathrm{c}}(y_0)), \tag{4.10a}$$

$$l_x = l^0_x + \int_0^t [c_{\mathrm{g}x}(k_{\mathrm{c}} - \Delta k) - c_{\mathrm{g}x}(k_{\mathrm{c}} + \Delta k)]\,\mathrm{d}t, \quad l^0_x = \frac{2\pi}{\Delta k^0_x} \approx \frac{2\pi}{\Delta\omega}\, c^0_{\mathrm{g}x}(k_0). \tag{4.10b}$$

Here k_{c} is the central instantaneous wavenumber of the packet, and Δk is the instantaneous halfwidth of the spatial spectrum. As a rule, the dispersive effects are dominating at small times, which results in a power decrease in the wave amplitude.

The competition between dispersive and focusing factors results in a complex law of wave-action evolution, which depends on a concrete relation between the packet and the inhomogeneity parameters. Let us restrict our consideration to the asymptotic laws of the packet-parameter evolution for large times. It is easily seen that the transverse dimension l_y tends to the limiting value $l^{\infty}_y \approx 2\,\Delta\omega\, D/N_{\mathrm{m}}$; each harmonic is trapped by its own layer of trapping. The longitudinal packet dimension l_x grows in two qualitatively different ways, depending on the type of the horizontal inhomogeneity responsible for the trapping. When the variation in N is a dominating factor, l_x also tends to the finite limit l_x^∞ ($l_x^\infty = \frac{1}{2} l_y^\infty \sim \Delta\omega\, D/N_m$). In the majority of real situations, however, the horizontal flow gradient is predominant. In this case, the longitudinal dimension of the packet grows linearly with time owing to the fact that the layer of trapping has a different y-coordinate for different harmonic components, and consequently moves at a different longitudinal velocity $U(y)$:

$$l_x^\infty = \tfrac{1}{2} l_y^\infty + U'_y\, l_y^\infty\, t \approx \frac{\Delta\omega}{N_m}(D+2Ut).$$

Having derived from (4.9) the asymptotic behaviour of $\bar{I}$ at large t, we can modify the asymptotic laws of wave-parameter variation (3.17), taking the dispersive spreading of the packets into account. In the first case ($N_m = N_m(y), k_x^0(y_0) = 0$), the numerical coefficient in the laws of growth decreases by a factor $\sim l_y^\infty / l_y^0$:

$$\frac{l_y^\infty}{l_y^0} \sim \frac{\Delta\omega^2}{N_m}\frac{D}{c_g^0} \sim \left(\frac{\Delta\omega}{N_m}\right)^2 kD.$$

In the second case ($U = U(y)$, $k_x^0(y_0) \neq 0$) there are two intermediate asymptotic forms. First, a law similar to that of the first case is realized, and then, at significantly larger times, the growth rate of the amplitude and of the wave steepness decreases by $\frac{1}{2}$ along t and by $-\frac{1}{2}\gamma$ along y.

4.3. *The simultaneous effect of viscosity and dispersion*

The simultaneous effect of viscosity and dispersive spreading can be taken into account with the same accuracy as in §4.2. Let us take the asymptotic laws of wave-parameter evolution modified by the bandwidth of the packet spectrum. We restrict our consideration to the case where the flow variability is the predominant factor:

$$A \sim (Nt)^{\frac{1}{2}+\frac{1}{4}\gamma} \exp\left(-\tfrac{2}{3}\nu \frac{N_m^2}{D^2} t^3\right), \tag{4.11a}$$

$$A_u \sim (Nt)^{\frac{1}{2}+1/\gamma-1/2\beta} \exp\left(-\tfrac{2}{3}\nu \frac{N_m^2}{D^2} t^3\right). \tag{4.11b}$$

Here A is the amplitude of the wave displacement and A_u is the amplitude of the horizontal velocity u (see figure 4*c*). The relations $kd \sim (y/D)^{-\gamma} \sim tN_m\, d/D$ are also valid here. Let us now estimate the maximum of the parameter A that corresponds to $k_{max}^{(A)} \sim (N_m/\nu D)^{\frac{1}{3}}$. In the parameter range considered in §4.1 the dispersive spreading (for $\Delta\omega/\omega \sim 10^{-1}$) leads to a decrease in A_{max}/A_0 by a factor varying from 10 to 10^7. Note that, when dispersion is not taken into account, $A_{max}/A_0 \sim 10$–10^2. The value A_{max} depends to a greater extent on the initial wavenumber k_0 and on the characteristic scale D of the flow inhomogeneity. The maximum of the steepness $(Ak)_{max}$ can either exceed unity at small k_0 and D (in this case a linear theory is not applicable, and the wave-breaking is likely to take place) or be much less than unity. In the latter case, a linear regime of viscous damping may be realized. In order to obtain an exact criterion for the initial steepness at which waves with a given k^0 will break at a given stratification and inhomogeneity, it is necessary to know the nonlinear laws of wave-parameter variation and the exact local criterion of breaking. Nevertheless, we can assume that long-enough waves trapped by a strong current with a large transverse gradient are most likely to break, forming turbulent paths.

4.4. *The applicability conditions for the adiabatic WKB approximation*

The results of the present paper have been obtained within the framework of the adiabatic WKB approximation. Let us analyse the boundaries of its applicability.

Far from the layer of trapping, the corrections stipulated by the right-hand side of (2.5) are small owing to the fact that kD is small, i.e. owing to the smoothness of variations in $U(y)$ and $N(y)$. In the vicinity of the layer of trapping, the values k and B, as well as the right-hand side of (2.3), tend to infinity, and the applicability of the adiabatic approximation should be substantiated separately.

It is known that a sufficient condition for the adiabatic WKB approximation to be applied in a certain region is the satisfaction of the following inequality (see e.g. Ginzburg 1961)

$$\frac{|n' \ln n|}{k_0 n^2} \ll 1, \tag{4.12}$$

where $n = n(y)$ is the index of refraction ($n = \omega_0 k/\omega k_0$) and $n' = \partial_y n$. At large k, i.e. in the vicinity of the layer of trapping of interest, $n \approx k/k_0$. This allows us to rewrite (4.12) in the form

$$\frac{|\partial_y k| \ln (k/k_0)}{k^2} \ll 1. \tag{4.13}$$

Substituting $k(y)$ for the stratification model (3.15) ($kd \sim (\alpha y)^{-\gamma}$) into (4.13), we can rewrite (4.13) as $(kd)^{1/\gamma - 1} \ln (k/k_0) \ll 1$ or

$$(\alpha y)^{\gamma-1} \ln (\alpha y)^{-\gamma} \ll 1. \tag{4.14}$$

It is clear that for $\gamma > 1$ the adiabatic WKB approximation remains valid up to the layer of trapping. At $\gamma \leqslant 1$ it seems necessary to verify whether (4.14) is valid for a viscous scale $k \sim k^A_{max} \sim (N_m/\nu D)^{\frac{1}{3}}$. For smooth profiles ($\gamma = 1$) the condition of the WKB applicability reduces to

$$\frac{\ln (k/k_0)}{k_0 D} \ll 1.$$

It is clear that for typical scales of oceanic internal waves this condition is always fulfilled.†

5. Discussion

Let us discuss the results from the point of view of their application to the description of complicated internal-wave dynamics in the upper ocean.

5.1. *Space–time scales*

The trapping of internal-wave modes may be caused by both the horizontal inhomogeneity of stratification and the horizontal velocity shear. A wave with the initial frequency ω_0 is trapped by a layer with the maximum effective Brunt–Väisälä frequency ($N_{ef} = N(z)/(1-kU/\omega_0)$) equal to ω_0. When the flow has no vertical shear

† The behaviour of the internal-wave field in an ideal fluid in the immediate vicinity of the layer of trapping, where the WKB approximation is no longer applicable, is also of interest. The investigation of this problem is far beyond the framework of this paper. Let us only note that the wave-field singularity is very strong and cannot be described by the usual analytic methods for caustics, since the group velocity, as well as *all* its derivatives, tend to zero in a singular point.

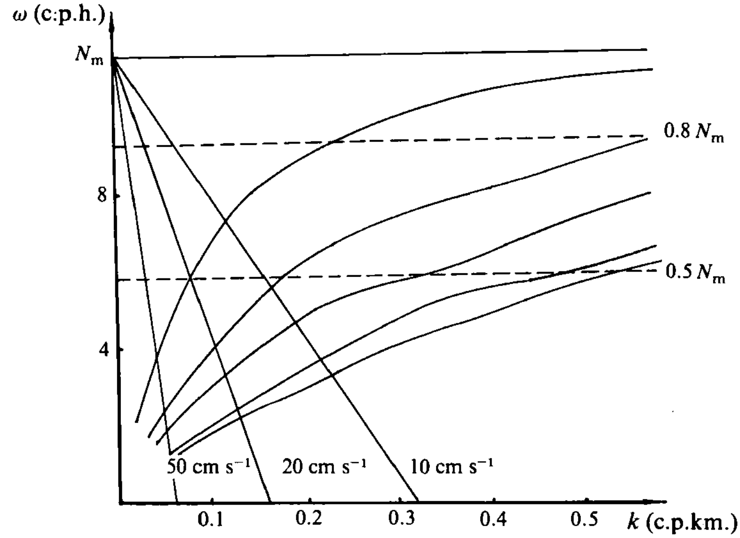

FIGURE 5. An example of the graphical finding of the wavenumber and frequency threshold values. The dispersion curves for the first four vertical modes presented in the figure are calculated for typical oceanic stratification (see figure 6). The threshold values k_t and ω_t are the points of intersection of dispersion curves by oblique straight lines at angle $-U_m$ in the case of inhomogeneous current, and points of intersection with horizontal dashed lines in the case of horizontal inhomogeneity of stratification.

($U_z = 0$), the condition of trapping reduces to the condition of equality between the Doppler-shifted frequency Ω and the local maximum $N(z, y)$.

Let us estimate the scales of the waves trapped by the flow with the maximum velocity U_m. They are determined from the condition

$$|k_x| > \frac{N_m - \omega_0(k)}{U_m}. \quad (5.1)$$

For a rough estimation we can use a simpler inequality $k > N_m/U_m$, then for $N_m \sim 10^{-3}\ \text{s}^{-1}$ and $U_m = 10$, 20, 50 cm s^{-1} we obtain the values for the threshold wavelengths $\lambda_t = 600$, 1200, 3×10^3 m respectively. A more precise estimate can be easily obtained graphically. As an illustration, let us take dispersion curves for the first four internal-wave modes (figure 5) calculated for a typical profile of stratification in the ocean (figure 6).† The threshold values of k and $\omega(k)$ are defined by the points of intersection of dispersion curves by oblique straight lines drawn from the point $(0, N_m)$ at an angle $-U_m$ to the abscissae axis. For the given example, $k_t = 12.5$, 8 and 5 c.p.km. for the first mode (figure 5) correspond to $U_m = 10$, 20 and 50 cm s^{-1}. The threshold wavenumbers for higher modes are considerably greater. This means that the waves of the first mode are trapped much easier.

When the trapping is caused by the horizontal inhomogeneity of stratification, the threshold wavenumbers are defined in a different manner. The condition of trapping

$$\omega(k) \geqslant N_{m\,min} \quad (5.2)$$

is fulfilled for short waves at relatively high frequencies ($\omega \sim N_m$). The threshold values k and ω are, graphically, the points of intersection of the dispersion curves

† Figures 5 and 6 have been kindly placed at our disposal by B. N. Filyushkin and V. V. Goncharov.

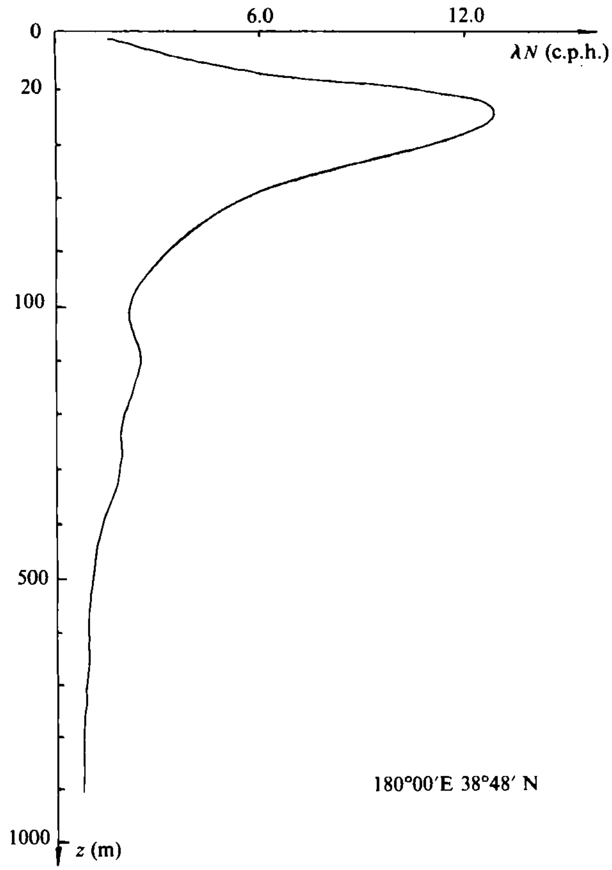

FIGURE 6. An example of a typical Brunt–Väisälä frequency profile.

plotted for the initial 'unperturbed' stratification $N^0(z)$ and of the horizontal straight lines with the ordinates $N_{\mathrm{m\,min}}$. For the case under consideration, the wavenumbers 20, 30 and 7 c.p.km. (for the first mode) correspond to $N_{\mathrm{m\,min}} = 0.8N^0_{\mathrm{m}}$ and $0.5N^0_{\mathrm{m}}$. Figure 7 (taken from Miropolsky, Solntseva & Filyushkin 1975) demonstrates the variations in the maximum Brunt–Väisälä frequency along the meridional cross-section. According to the data of this paper, the mean value of $\partial_y N_{\mathrm{m}}$ is of order $10^{-8}\ \mathrm{s}^{-1}\ \mathrm{m}^{-1}$, and the typical spacescale D of N_{m} variation is $\sim 10^6$ m. At the same time, the gradients of N_{m} increase up to $3\times10^{-7}\ \mathrm{s}^{-1}\ \mathrm{m}^{-1}$ in some regions (generally corresponding to the frontal zones), and the scale D decreases down to $\sim 10^4$ m.

Thus the trapping (mainly by the flow inhomogeneities) of waves with scales from tens to thousands of metres is possible at the typical values of the parameters.

5.2. *The main stages of the evolution*

As the wave packet approaches the layer of trapping, the wavenumber k grows. This growth is accompanied by the transformation of the vertical structure of internal-wave modes and is restricted only by the breaking of the waves and by viscous dissipation. At large k the transformation of the $w(z,k)$ modes has a universal character for all smooth dependences of $N(z)$ with a vertical variability scale d: $w(z) \sim \exp(-kz^2/2d)$. Thus the narrowing of the waveguide leads to vertical wave focusing, i.e. to concentration of the wave motion at the depth z_{m}, where the effectuve Brunt–Väisälä

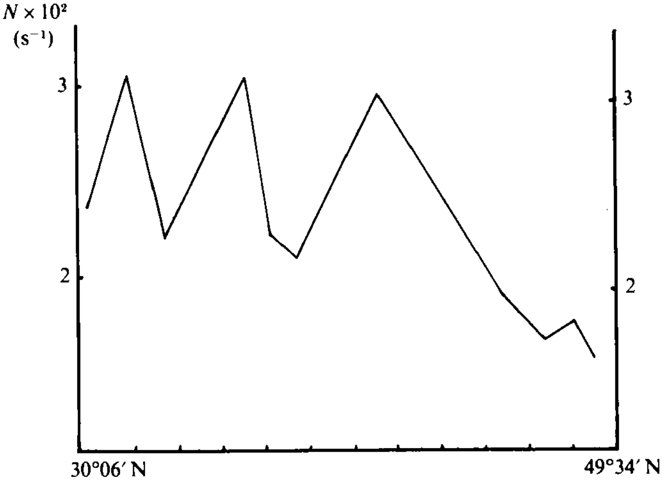

FIGURE 7. Meridional cross-section for N_{m}.

frequency N_{ef} is maximum. In the case of the vertically homogeneous flow $U(y)$, the focusing of all trapped waves occurs at the depth z_{m} that corresponds to N_{m}.

The trapping of the wave packet is accompanied by the growth of the wave amplitude and steepness. The growth of the amplitude with time is caused by two factors: the slowing down of the packet owing to a decrease in the wavelength, and the vertical focusing that leads to the accumulation of the wave energy in a narrow layer. The finite bandwidth of the packet spectrum stipulates its dispersive spreading that slows down the amplitude growth. At small times the dispersion may even lead to a decrease in the wave amplitude. Then an intermediate asymptotic behaviour is established: for a smooth profile $N(z)$ the amplitude of the wave displacement A grows as $t^{\frac{5}{4}}$ if the trapping is caused by the variation of N, and as $t^{\frac{3}{4}}$ if the flow inhomogeneity is a predominant factor. In this case the wave steepness and the parameter of nonlinearity grow by a factor of $(Nt)^{\frac{1}{2}}$ faster. The power laws of the amplitude and wave-steepness growth are valid not only for smooth profiles $N(z)$. The growth of the wave steepness is accompanied by an increased probability of wave-breaking. However, in the framework of the linear theory considered here, the saturation of the amplitude growth can be associated only with the viscous dissipation. At large times $(t \gg (D/N_{\mathrm{m}})^{\frac{2}{3}} \nu^{\frac{1}{3}})$ the packet motion slows down, and the viscous dissipation, although remaining small, becomes the predominant factor in the evolution. This leads to the damping of the packet amplitude $\sim \exp(-\nu N_{\mathrm{m}}^2 t^3/D^2)$, i.e. to the effective cutoff of small horizontal $(k \geqslant (\nu D/N_{\mathrm{m}})^{-\frac{1}{3}})$ and thus small vertical $(l_z \lesssim (\nu D/N_{\mathrm{m}})^{\frac{1}{6}} d^{\frac{1}{2}})$ scales.

The characteristic time τ of the internal-wave transformation into the viscous-dissipation range $(\tau \sim (D/N_{\mathrm{m}})^{\frac{2}{3}} \nu^{-\frac{1}{3}})$ does not, practically, depend on the initial wave parameters. It depends slightly on the coefficient of viscosity, but varies widely (from one to some scores of days) depending on the horizontal inhomogeneity scale D. The minimum spacescale of the waves that is determined by the viscous damping is of order $\sim 10^3$ cm and also weakly depends on viscosity. A regime of viscous damping seems to be possible at small-enough initial amplitudes of the waves.

5.3. *The effect of non-stationarity*

The ambient fields for which the basic equations were linearized were formerly considered to be stationary. However, when the timescales of the packet evolution are comparable to the period of low-frequency motions, which are always present in the ocean, the applicability of the conclusions formerly obtained for the final (slow) stage of the packet evolution in the real ocean needs substantiation.

Consider the motion of the wave packet in an inhomogeneous flow $U(y)$ in the presence of a long travelling wave. Since the group velocity of the packet is much smaller than the phase velocities of long waves, one can assume that the wave component of the ambient medium depends, mainly, on time. The packet dynamics is affected only by the horizontal velocity component of the ambient motion $\boldsymbol{U}$:

$$\boldsymbol{U} = U_0(y)\,\boldsymbol{i} + \tilde{U}(y,t)\,\boldsymbol{i} + \tilde{V}(y,t)\,\boldsymbol{j}.$$

The equations of motion of a spectrally narrow packet with a central frequency ω and a wavevector $\boldsymbol{k}(k_x, k_y)$ can be written in the form (Voronovich 1976*b*)

$$\dot{x} = c_{gx} + U_0 + \tilde{U}, \quad \dot{y} = c_{gy} + \tilde{V}, \tag{5.3}$$

$$\dot{k}_x = 0, \quad \dot{k}_y = k_x \partial_y (U_0 + \tilde{U}) + k_y \partial_y \tilde{V}_y. \tag{5.4}$$

Since the dependence of $\boldsymbol{U}$ on y is much slower than that on t, we obtain from (5.4)

$$k_y = k_y(t_0) \exp\left(-\int_{t_0}^{t} \partial_y \tilde{V}\,\mathrm{d}t\right) - k_x \int_{t_0}^{t} \partial_y (U_0 + \tilde{U})\,\mathrm{d}t. \tag{5.5}$$

Assuming $\tilde{U}$, $\tilde{V} \sim \sin(\kappa y - \sigma t)$ and $\kappa |U|/\sigma \ll 1$, we have from (5.5)

$$k_y = k_y^0 - k_y^0 \frac{\kappa}{\sigma} \tilde{V} + k_x \frac{\kappa}{\sigma} \tilde{U} - k_x \partial_y U_0 t. \tag{5.6}$$

It is clear from (5.6) that the low-frequency wave motions lead to periodic variation of the transverse wavenumber k_y with respect to its mean value, but do not affect the linear growth of its mean value with time. It also follows from (5.6) that the asymptotic behaviour of k_y ($k_y \sim k_x U_0 t/D$) is determined by a stationary velocity component.

Equations (5.3) and (5.4) show that the trajectory of the packet is a helix with almost elliptic turns. The trajectory of the turn centre does not depend on the nonstationary velocity component.

Thus low-frequency wave motions lead to an oscillatory (with a long wave period) *variation in the packet parameters, and do not affect the dynamics of parameter values averaged over the period.* This fact indicates that the results obtained for a stationary environment can be used for the description of wave-packet evolution in the real non-stationary ocean.

We shall avoid discussing the strong assumption accepted here of a purely transverse variability of the ambient flow velocity $U = U(y)$. The question naturally arises as to whether the effect of trapping is structurally stable with respect to the existing longitudinal variability of the flow velocity $\boldsymbol{U}(\boldsymbol{x})$. When the group-velocity component c_{gy} becomes equal to the transverse component V of the flow velocity, a packet blocking (Basovich & Tsimring 1984) may prevent the trapping. The results of the study of internal-wave dynamics on non-uniform three-dimensional currents of arbitrary form (including the study of trapping under these conditions) will be reported elsewhere. Here we only mention that the longitudinal variability of mean flow in the ocean is generally much weaker than the transverse one ($U_x \ll U_y$), and the characteristic values of the transverse flow velocity are often negligible.

5.4. *The role of trapping in the energy balance of internal waves and currents*

Let us now discuss a possible role of trapping in the energy exchange between internal waves and flows. Since the results obtained in the framework of the linear theory are the only ones available so far, this discussion will inevitably be a bit speculative.

At the inviscid stage of packet evolution the wave action I ($= E/\Omega$, where E is the wave-energy density and $\Omega = \omega - k_x U$ is the Doppler-shifted frequency) is an invariant. The variation of the frequency Ω caused by the flow gradient stipulates the variation of the wave-energy density E:

$$E(y) = E_0(y_0)\,\frac{\Omega(y)}{\Omega_0(y_0)}, \quad E - E_0 = E_0\left(\frac{\Omega}{\Omega_0} - 1\right).$$

As the packet approaches the layer of trapping, Ω increases monotonically from Ω_0 to N_m, so that the *wave takes energy from the flow at the inviscid stage*. The maximum possible energy acquired by the wave (if its breaking has not already occurred) is equal to $E_0(N_m/\Omega_0 - 1)$. The total energy loss of the flow can be estimated by the energy spectrum of the waves incident to the flow.

As the packet approaches the layer of trapping, significant increases in the wavenumber and amplitude occur, i.e. the energy of relatively long internal waves and the energy taken from the flow is transferred to small-scale internal waves. Further, in a linear theory this energy is transformed directly into heat owing to viscous dissipation (part of the energy can be transferred to the mean flow). The sink of the internal-wave energy into heat via small-scale turbulence due to the wave-breaking (the possibility of breaking increases with a growth in the wave steepness) is probably an essential component of the internal-wave-field energy sink. In this strongly nonlinear mechanism, the internal waves play the role of an intermediate link in the energy exchange between large-scale motions and small-scale turbulence. When only the mechanisms of viscous dissipation and wave-breaking are active, the energy of all trapped waves is taken from the internal-wave field. Then the neighbourhood of the layers of trapping is the region of the internal-wave energy sink. An estimation of the internal-wave energy sink carried out for the Garrett–Munk spectrum with realistic inhomogeneity parameters gives us a value that greatly exceeds the known estimates. This indicates indirectly an important role of another nonlinear mechanism in the energy balance – the mechanism of the nonlinear interaction between the trapped packet and the internal-wave ambient field. Depending on the intensity of nonlinear interactions, the neighbourhood of the layers of trapping can be both the region of an intensive internal-wave energy sink and the generation region. However, the role of trapping in the energetics of the upper ocean can be clarified only when the nonlinear stage of trapping is studied.

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